Cvičení 5 (limita funkce)   Opakování: 18.10.22

Finta 1: vytkni nejvyšší mocninu
Finta 2: vytkni exp o nejvyšším základu
Finta 3: rozšiř znaménkem opačným

Př. 1. $\lim\limits_{n\to+\infty} \dfrac{3\cdot 5^n - 2^n}{\left(\frac{5}{2}\right)^n + 1} = \lim\limits_{n\to+\infty} \dfrac{5^n\left[3 - \left(\frac{2}{5}\right)^n\right]}{5^n\left[\left(\frac{1}{2}\right)^n + \frac{1}{5^n}\right]} =$

$= \lim\limits_{n\to+\infty} 2^n \dfrac{3 - \left(\frac{2}{5}\right)^n}{1^n + \frac{1}{(2.5)^n}} \overset{\text{VoAL}}{=} \underbrace{\lim\limits_{n\to+\infty} 2^n}_{\to+\infty} \cdot \lim\limits_{n\to+\infty} \dfrac{3 - \overbrace{\left(\frac{2}{5}\right)^n}^{\to 0}}{\underbrace{1^n}_{1} + \underbrace{\frac{1}{10^n}}_{0}} = +\infty$

nevlastní limita diverguje.

jinak bychom dostali nedefinovaný výraz $K/0$
const.

Př. 2. $\lim\limits_{n\to+\infty} \dfrac{\left(\frac{3}{2}\right)^{2n} + \left(\frac{7}{4}\right)^{n+1}}{\left(\frac{3}{2}\right)^{n+1} + \left(\frac{9}{4}\right)^{n+1}} \overset{\text{dosazení}}{=} \text{"}\dfrac{+\infty}{+\infty}\text{"}$ nedefinovaný výraz $\left(\frac{3}{2}\right)^{2n} = \left(\frac{9}{4}\right)^n$

$= \lim\limits_{n\to+\infty} \dfrac{\left(\frac{9}{4}\right)^n + \left(\frac{7}{4}\right)^{n+1}}{\left(\frac{3}{2}\right)^{n+1} - \left(\frac{9}{4}\right)^{n+1}} \overset{(F2)}{=} \lim\limits_{n\to+\infty} \dfrac{\cancel{\left(\frac{9}{4}\right)^n}}{\cancel{\left(\frac{9}{4}\right)^n}} \dfrac{1 + \left(\frac{7}{4}\right)^{n+1}\cdot\left(\frac{4}{9}\right)^n}{\left(\frac{3}{2}\right)^{n+1}\left(\frac{4}{9}\right)^n - \frac{9}{4}} =$

$\left(\frac{3}{2}\right)^n\left(\frac{3}{2}\right)\left(\frac{4}{9}\right)^n = \left(\frac{3}{2}\right)^n \cdot \left(\frac{3}{2}\right)\left(\frac{2}{3}\right)^{2n} = \left(\frac{2}{3}\right)^n \frac{3}{2}$

$= \lim\limits_{n\to+\infty} \dfrac{1 + \left(\frac{7}{9}\right)^n \cdot \frac{7}{4}}{-\frac{9}{4} + \left(\frac{2}{3}\right)^n \frac{3}{2}} =$

$\overset{\text{VoAL}}{=} \dfrac{\lim\limits_{n\to+\infty} 1 + \overbrace{\left(\frac{7}{9}\right)^n}^{\to 0} \cdot \frac{7}{4}}{\lim\limits_{n\to+\infty} -\frac{9}{4} + \underbrace{\left(\frac{2}{3}\right)^n}_{\to 0} \frac{3}{2}} = -\dfrac{4}{9}$

Připomenutí:
$a^2 - b^2 = (a-b)(a+b)$

vedle pak nedefinovaný výraz

Př. 3 $\lim\limits_{n\to+\infty} \sqrt{n}\left(\sqrt{n+2} - \sqrt{n}\right) \overset{(F3)}{=} \lim\limits_{n\to+\infty} \sqrt{n}\left(\sqrt{n+2} - \sqrt{n}\right) \cdot \dfrac{\sqrt{n+2} + \sqrt{n}}{\sqrt{n+2} + \sqrt{n}} =$

$= \lim\limits_{n\to+\infty} \dfrac{\sqrt{n}}{\sqrt{n+2} + \sqrt{n}} \cdot \left[(n+2) - n\right] = \lim\limits_{n\to+\infty} \dfrac{2\sqrt{n}}{\sqrt{n+2} + \sqrt{n}} \Rightarrow \text{"}\dfrac{+\infty}{+\infty}\text{"}$, třeba použít (F1)

$\overset{(F1)}{=} \lim\limits_{n\to+\infty} \dfrac{\sqrt{n}}{\sqrt{n}} \dfrac{2}{1 + \sqrt{1 + \frac{2}{n}}} \overset{\text{VoAL}}{=} \dfrac{\lim\limits_{n\to+\infty} 2}{\lim\limits_{n\to+\infty} 1 + \sqrt{1 + \underbrace{\tfrac{2}{n}}_{\to 0}}} = \dfrac{2}{2} = 1$

Př. 4 $\lim\limits_{n\to+\infty} \sqrt{n^2-n-1} - \sqrt{n+1} \overset{(F3)}{=}$

(9:15)

$$\lim_{n\to+\infty}\left(\sqrt{n^2-n-1}-\sqrt{n+1}\right)\frac{\sqrt{n^2-n-1}+\sqrt{n+1}}{\sqrt{n^2-n-1}+\sqrt{n+1}} = \lim_{n\to+\infty}\frac{n^2-n-1-(n+1)}{\sqrt{n^2-n-1}+\sqrt{n+1}} =$$

$$= \lim_{n\to+\infty}\frac{n^2-2n-2}{\underbrace{\sqrt{n^2-n-1}}+\underbrace{\sqrt{n+1}}} \overset{(F1)}{=} \lim_{n\to+\infty}\frac{n^2\left(1-\frac{2}{n}-\frac{2}{n^2}\right)}{n\sqrt{1-\frac{2}{n}-\frac{1}{n^2}}+n\sqrt{\frac{1}{n}+\frac{1}{n^2}}} =$$

$$= \sqrt{n^2\left(1-\frac{2}{n}-\frac{1}{n^2}\right)} \qquad = \sqrt{n^2\left(\frac{1}{n}+\frac{1}{n^2}\right)}$$

$$\overset{VAAL}{=} \lim_{\substack{n\to+\infty\\ \searrow +\infty}} n \cdot \lim_{n\to+\infty}\frac{1-\overset{\nearrow 0}{\frac{2}{n}}-\overset{\nearrow 0}{\frac{2}{n^2}}}{\sqrt{1-\underset{\downarrow 0}{\frac{2}{n}}-\underset{\downarrow 0}{\frac{1}{n^2}}}+\sqrt{\underset{\downarrow 0}{\frac{1}{n}}+\underset{\downarrow 0}{\frac{1}{n^2}}}} = +\infty$$

vlastní limita, diverguje.

Př. 5 $\lim\limits_{n\to+\infty}\dfrac{\sqrt{n^2+n+1}-\sqrt{n^2-1}}{\sqrt{2n-12}} \overset{(F3)}{=} \lim\limits_{n\to+\infty}\dfrac{\sqrt{n^2+n+1}-\sqrt{n^2-1}}{\sqrt{2n-12}}\cdot\dfrac{\sqrt{n^2+n+1}+\sqrt{n^2-1}}{\sqrt{n^2+n+1}+\sqrt{n^2-1}} =$

$$= \lim_{n\to+\infty}\frac{(n^2+n+1)-(n^2-1)}{\left(\sqrt{2n+12}\right)\left(\sqrt{n^2+n+1}+\sqrt{n^2-1}\right)} = \lim_{n\to+\infty}\frac{n+2}{\left(\sqrt{2n-12}\right)\left(\sqrt{n^2+n+1}+\sqrt{n^2-1}\right)} =$$

$$= \lim_{n\to+\infty}\frac{n+2}{\underbrace{\sqrt{(2n-12)(n^2+n+1)}}+\underbrace{\sqrt{(2n-12)(n^2-1)}}} \overset{(F1)}{=} \lim_{n\to+\infty}\frac{n\left(1+\frac{2}{n}\right)}{n^{3/2}\left[\left(2-\frac{10}{n}-\frac{10}{n^2}-\frac{12}{n^3}\right)^{1/2}+\left(2-\frac{12}{n}-\frac{2}{n^2}+\frac{12}{n^3}\right)^{1/2}\right]}$$

$$= \sqrt{2n^3-10n^2-10n+12} \qquad = \sqrt{2n^3-12n^2-2n+12} \qquad \frac{n}{n^{3/2}} = \frac{1}{\sqrt{n}}$$

$$\overset{VAAL}{=} \lim_{\substack{n\to+\infty\\ \searrow 0}}\frac{1}{\sqrt{n}}\lim_{n\to+\infty}\frac{1+\frac{2}{n}\to 0}{\left[\left(2-\underset{\downarrow 0}{\frac{10}{n}}-\underset{\downarrow 0}{\frac{10}{n^2}}-\underset{\downarrow 0}{\frac{12}{n^3}}\right)^{1/2}+\left(2-\underset{\downarrow 0}{\frac{12}{n}}-\underset{\downarrow 0}{\frac{2}{n^2}}+\underset{\downarrow 0}{\frac{12}{n^3}}\right)^{1/2}\right]} = 0$$

vlastní limita, konverguje

(zkouškový limita)

Limita funkce

- od posloupností přecházíme k funkcím

$$\lim_{x \to x_0} f(x) = A \quad \begin{cases} A \in \mathbb{R} & \text{vlastní limita} \\ A \in \pm\infty & \text{nevlastní limita} \end{cases} \quad \begin{cases} x_0 \in \mathbb{R} & \text{vlastní bod} \\ x_0 \in \pm\infty & \text{nevlastní bod} \end{cases}$$

„limita funkce v bodě x_0"

$$\left.\begin{matrix} \lim_{x \to x_0^+} f(x) \\ \lim_{x \to x_0^-} f(x) \end{matrix}\right\} \text{jednostranné limity}$$

POKUD $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x)$, potom

$$\exists \lim_{x \to x_0} f(x)$$

Q: Jak souvisí spojitost s limitami?

Funkce $f(x)$ je spojitá v $x=a$, pokud jsou splněny tyto tři podmínky:

(1) $\lim_{x \to a} f(x)$ existuje

(2) $f(a)$ existuje

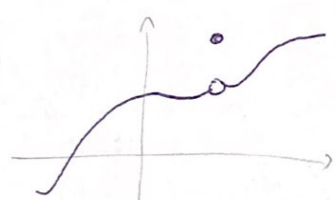
(3) $\lim_{x \to a} f(x) = f(a)$

Př: $h(x) = \frac{x^2-4}{x^2+x-2}$

odstranitelná nespojitost

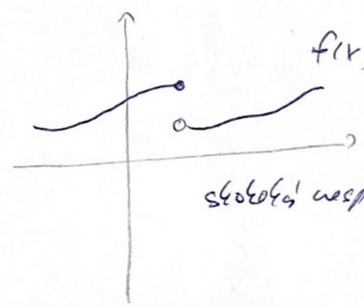
Q: Jaké typy nespojitostí známe?

a)



bodová nespojitost

c)



skoková nespojitost

odstranitelná nespojitost

↳ funkci lze v bodě nespojitosti dodefinovat limitou, pokud

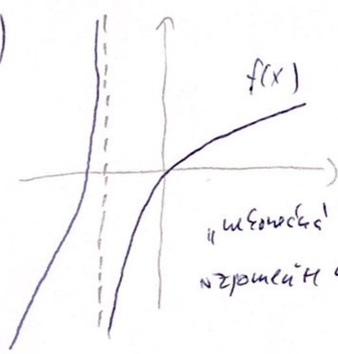
$$A := \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x),$$

která je však různá od $f(a)$

Tzn.

$$g(x) \begin{cases} \lim_{a \to a} f(x) & x = a \\ f(x) & x \neq a \end{cases}.$$

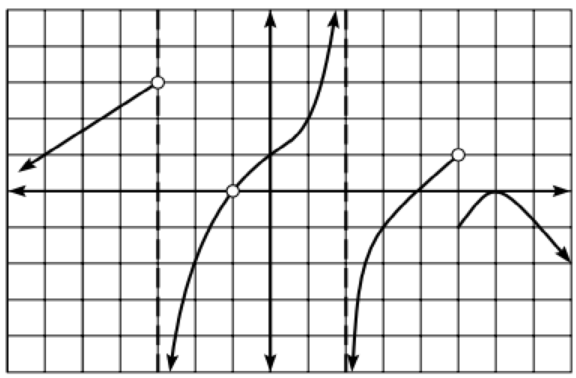
b)



„nekonečná" nespojitost

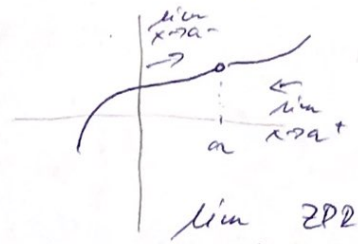
vzpomeňte na asymptoty

Př:



$p(x)$

Jednostranné limity:



$\lim_{x \to a^+}$ ZPRAVA

$\lim_{x \to a^-}$ ZLEVA

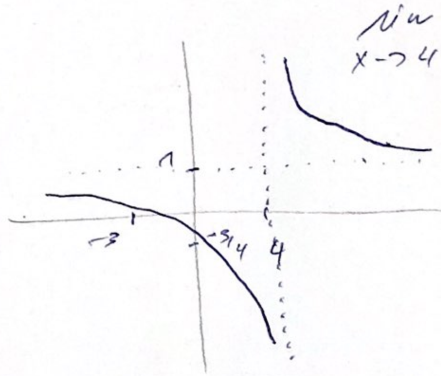
1. $\lim_{x \to 2^-} p(x) = +\infty$ nevlastní limita
2. $\lim_{x \to 3^+} p(x) = -1$ $\quad \lim_{x \to 3^-} p(x) = -1$
3. $\lim_{x \to 3} p(x) = -1$, protože $\lim_{x \to 3^+} = \lim_{x \to 3^-}$
4. $\lim_{x \to 5^-} p(x) = 1$ $\Big\}$ $\lim_{x \to 5}$ neexistuje
5. $\lim_{x \to 5^+} p(x)\ -1$
6. $\lim_{x \to -1} p(x) = 0$ (funkce v bodě -1 není definovaná, ale $\lim_{x \to -1^+} = \lim_{x \to -1^-} = 0$)

Limity racionálních l. funkcí.

$\lim_{x \to 4^+} f(x)\ +\infty$
$\lim_{x \to 4^-} f(x) = -\infty$ $\Big\}$ $\lim_{x \to 4}$ neexistuje

Př: $f(x) = \frac{x+3}{x-4}$

$D_f = \mathbb{R} \setminus \{4\}$



$\lim_{x \to 6^+} f(x) = 9/2$

$\lim_{x \to 6^-} f(x) = 9/2$ existuje

$(x+3):(x-4) = 1 + \frac{7}{x-4}$

$-(x-4)$

$\overline{\quad 7 \quad}$

$S = [4, 1]$

$P_x: y = 0 \quad P_x = [-3, 0]$

$P_y: x = 0 \quad P_y = [0, -3/4]$

$\lim_{x \to +\infty} \frac{x+3}{x-4} = \lim_{x \to +\infty} \frac{x(1 + 3/x)}{x(1 - 4/x)} = 1$

$\lim_{x \to -\infty} \frac{x+3}{x-4} = \lim_{x \to -\infty} \frac{x}{x} \cdot \frac{1 + 3/x}{1 - 4/x} = 1$

Pr. $f(x) = \frac{5x^2 - x}{x^2 - 4}$ $\lim_{x \to \pm\infty} \frac{5x^2 - \cancel{x}}{x^2 - 4} = \lim_{x \to +\infty} \frac{x^2}{x^2} \frac{5 - 1/x}{1 - 4/x^2} =$

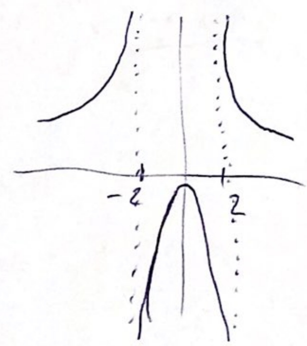
$D_f = \mathbb{R} \setminus \{\pm 2\}$ $= 5$ //

$\lim_{x \to \pm 2^{\pm}}$

$\lim_{x \to 2^+} \frac{5x^2 - x}{x^2 - 4} = \frac{5 \cdot (2^2) - 2}{0^+} = \frac{18}{0^+} = +\infty$

$\lim_{x \to 2^-} \frac{5x^2 - x}{x^2 - 4} = \frac{5 \cdot (2)^2 - 2}{0^-} = -\infty$ $\lim_{x \to -2^-} (\;) = +\infty$

$\lim_{x \to 2^+} \frac{5x^2 - x}{x^2 - 4} = \frac{5 \cdot (-2)^2 + 2}{0^-} = \frac{22}{0^-} = -\infty$



~~Pr. f(x) =~~ $\cancel{\frac{5x^3 + 5x + 2}{6x^3 - 3x^2}}$

~~$6x^3 - 3x^2 = x^2(6x - 3)$~~

~~Kritične točke: $x \neq 0$, $x \neq 1/2$, $\pm\infty$~~

~~$\lim_{x \to \pm\infty} \frac{5x^3 + 5x + 2}{6x^3 - 3x^2} = \frac{5}{6}$~~

~~$\lim_{x \to 0^+}$~~